EINIGE GESETZE ÜBER DIE THEILUNG DER EBENE UND DES RAUMES.

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ABSTRACT. The article "Einige Gesetze über die Theilung der Ebene und des Raumes." was published by J. Steiner in the very first volumn of the *Journal für die reine und angewandte Mathematik* in 1826. Journal für die reine und angewandte Mathematik is the oldest mathematics periodical in existence.[†]. My translation is meant to convey the ideas published by Steiner, and when presented with the choice between translating faithfully to the original text or clarity of his ideas, I admit to choosing the later. There are two footnotes original to the text, which appear with asterisks. Where helpful I included additional footnotes to clarify ideas in the article; these are denoted by daggers and do not appear in the original text. I welcome any corrections or improvements to this translation. Please contact me with suggestions at heavilin@usu.edu

Geometry textbooks have primarily followed J. Pestalozzi's methodology[†] to illustrate how many planer regions can be constructed from intersections of lines and circles. But this treatment does not reveal the underlying rules that permit a more general treatment of the topic. Moreover, the approach is even less effective in providing general rules for the formation of regions in space formed by intersections of arbitrary planes and spherical surfaces.

Instead of the geometer's question, "How many flat surfaces are necessary to create this or that body?", we turn the question around and ask, "How many regions can be formed by a given number of planes?" We begin by observing at least four planes are needed to construct a bounded region in space. By extension conclude four planes can form at most one bounded region. This statement then begs the question, "How many regions can be formed by $4, 5, 6, 7$, or more planes?"

In this article we present general rules for dividing a plane into bounded regions by way of intersecting lines and circles. Thereafter we extend these ideas to space and present the rules determining the number of regions formed by intersecting planes and spherical surfaces. From this we can easily determine how many of these regions are bounded. Unlike the current treatment of this topic in geometry textbooks, we explain the relevant relationships and provide a context for answering these questions, and in doing so further the study of solid geometry.

1

Clearly a straight line[*] cut at n arbitrary points is broken into $n+1$ regions[**] of which $n-1$ are finite and the two remaining regions are infinite, and that furthermore a closed-curve[††] cut at n points is broken into n regions.

2

A straight line embedded in a plane cuts the plane into two regions; with the addition of a second line that cuts the first the number of regions of the plane increases by two. A third line that cuts the first lines, the number of regions increases to three, and with a fourth line, that cuts the first three in three points, to four, and so on. In fact each susequent line increases the number of regions in the plane by the number of existing regions through which the new line passes, therefore the plane with n lines will be cut into

$$\begin{aligned} 2+2+3+4+5+\cdots+(n-1)+n &= 1+\frac{n(n+1)}{2} \\ &= 1+n+\frac{n(n-1)}{1\cdot 2} \qquad (1) \end{aligned}$$

[†]Johann Heinrich Pestalozzi (January 12, 1746 Ð February 17, 1827) was a Swiss pedagogue and educational reformer who exemplified Romanticism in his approach. - Wikipedia

[*]By straight line we mean an infinite line.

[**]Also here we mean the following, when speaking of the regions of a plane or space, we mean only the single regions, and not regions constructed by adjoining more simple regions together.

[††]Here the author is referring to circles

regions.

If one simply wants to know the number of finite (bounded) regions of the plane, we note that the first three lines construct this piece. The fourth line increases the number of such regions by 2 and the fifth line increases by 3 and so on. Namely, for each additional line the number of bounded regions of the plane can increase by the number of intersections with boarders of the bounded regions that it interesects†, and therefore with n arbitrary lines there can be at most

$$\begin{aligned} 0+0+1+2+3+4+\cdots+(n-3)+(n-2) &= \frac{(n-1)(n-2)}{2} \\ &= 1-n+\frac{n(n-1)}{1\cdot 2} \end{aligned} \tag{2}$$

bounded regions of the plane.

Substracting Eq. (2) from Eq. (1), the number of unbounded regions of the plane is

$$2n. \tag{3}$$

Alternatively, if we find the number of unbounded regions, which we see increases by 2 with each additional line, and consequently the number is $2n$, then we can determine the number of finite regions by subtracting $2n$ from Eq. (1).

3

With a parallel lines the plane will be cut into $1+a$ regions, with a second partitioning from b parallel lines, each intersecting the first a lines, the number of regions increases to $b(1+a)$. A third partitioning from c parallel lines cutting the previous sets of lines in such a way that three lines never meet at the same point, the number of regions increases to $c(1+a+b)$. If one includes d parallel lines, under the same conditions, then the number of regions is $d(1+a+b+c)$, in which each of the lines from the existing lines a, b, c is increased by $1+a+b+c$ and so on. It follows from this that,

Rule 3.1: *Sets of parallel lines $a, b, c, \cdots, y, z$, where each set is pointing in a different direction, can partition the plane into at most*

$$\begin{aligned} 1+a& \\ +b(1+a)& \\ +c(1+a+b)& \\ +d(1+a+b+c)& \\ \vdots\ & \\ +z(1+a+b+c+\cdots y) &= 1 \\ &\quad +a+b+c+d+\cdots+y+z \\ &\quad +ab+ac+\cdots+bc+bd+\cdots+yz \end{aligned} \tag{4}$$

regions.

If we let the sum (Union) of the quantities $a, b, c, d, \cdots, y, z$ be U, and the sum of pairwaise products (Amben) by A, then Eq. (4) is

$$= 1+U+A \tag{5}$$

Just as in the result from §2, where the number of unbounded regions increases by 2 with the addition of each line, so too in this case the number of unbounded regions of the plane is twice the total number of existing lines. Or consider a circle that intersects all of the previous sets of paralllel lines and in such a way that all intersection points lie inside the circle. Then the circle is cut into as many regions as there are unbounded regions of the plane.

†Alternately, by the number of finite line segments intersected by the new line.

Now since the circle is cut twice by each line, it will be cut into twice as many regions as the number of existing lines (see §1), i.e. into $2U$ regions, and it follows that the number of unbounded regions of the plane is

$$= \quad 2U. \tag{6}$$

We can get the number of finite regions by subtracting the number of all unbounded regions (6) from the total number of regions (5). So by the aforementioned construction, the number of bounded regions is at most

$$= \quad 1 - U + A \tag{7}$$

Equation (7) can also be found in the same manner as Eq. (5) or as in §2.

4

Consdier that the first set of lines in §3 are running in different directions instead of being parallel. Then they cut the plane into $1 + \dfrac{a(a+1)}{2}$ regions, instead of $1 + a$, and hence into $\dfrac{a(a-1)}{2}$ more regions than when those lines are parallel, but this does not change the partitioning of the others. It therefore follows:

Rule 4.1: *That the plane cut by $b, c, d, \cdots, y, z$ sets of paralell lines pointing in different directions and with a non-parallel lines partition the plane into at most*

$$= \quad 1 + U + A + \frac{a(a-1)}{1 \cdot 2} \tag{8}$$

of which

$$= \quad 2U \tag{9}$$

are unbounded, and therefore

$$= \quad 1 - U + A + \frac{a(a-1)}{1 \cdot 2} \tag{10}$$

bounded regions, where U and A are as defined in §3, the sum and pairwise-product of the magnitudes $a, b, c, d, \cdots, y, z$.

5

A circle cuts the plane into two regions. A second circle, that intersects the first, increases the number of regions by 2. A third circle, that intersects both the first two circles at 4 points, increases this number by 4, and so on. Namely, each susequent circle increases the number of regions by as much as the number of intersection points wiht the existing circles, i.e. twice the number of existing circles. It therefore follows that

Rule 5.1: $\mathfrak{n}$ *arbitrary circles cut the plane into at most*

$$2 + 2 + 4 + 6 + 8 + \cdots + 2(\mathfrak{n} - 1) \quad = \quad 2 + \mathfrak{n}(\mathfrak{n} - 1) \tag{11}$$

regions, of which

$$= \quad 1 + \mathfrak{n}(\mathfrak{n} - 1) \tag{12}$$

are bounded and only one piece is unbounded.

6

Any number, $\mathfrak{a}$, of concentric circles will cut the plane into $1 + \mathfrak{a}$ regions; with a second partitioning of $\mathfrak{b}$ concentric circles intersecting the first circles, will increase the number of regions by $1 - \mathfrak{a} + 2\mathfrak{a}\mathfrak{b}$ (namely the first by $1 + \mathfrak{a}$ and then each following by $2\mathfrak{a}$); with a third partitioning from $\mathfrak{c}$ concentric cicles, that intersect the first, the number will increase by $2\mathfrak{c}(\mathfrak{a} + \mathfrak{b})$, through a fourth partitioning of $\mathfrak{d}$ concentric circles the number will be increased by $2\mathfrak{d}(\mathfrak{a} + \mathfrak{b} + \mathfrak{c})$ and so on.

Namely, each circle increases the number of regions by exactly as many regions created by the intersecting circles. (of these only the first circle of the second set is excluded.)
Consequently,

Rule 6.1: *The plane cut by* $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{b}, \dots \mathfrak{y}, \mathfrak{z}$ *sets of concentric cicles, of which has a distinct center point, cuts the plane into at most:*

$$\begin{array}{lll} & 1+\mathfrak{a} & \\ & 1-\mathfrak{a} & +2\mathfrak{a}\mathfrak{b} \\ & & +2\mathfrak{c}(\mathfrak{a}+\mathfrak{b}) \\ & & +2\mathfrak{d}(\mathfrak{a}+\mathfrak{b}+\mathfrak{c}) \\ & & \vdots \\ & & +2\mathfrak{z}(\mathfrak{a}+\mathfrak{b}+\mathfrak{c}+\mathfrak{d}+\cdots+\mathfrak{y}) \\ (13) & = & 2+2\,\mathfrak{A} \end{array}$$

regions, of which

$$(14) \qquad = \quad 1+2\,\mathfrak{A}$$

are bounded and only one is unbounded, and where $\mathfrak{A}$ *is the Amben, i.e. the sum of all pairwise products when each pair from* $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{b}, \dots \mathfrak{y}, \mathfrak{z}$ *is multiplied by the other.*

7

If the $\mathfrak{a}$ circles from the first partition are not concentric, then the number of regions thereby increase at most as the number of intersections of these $\mathfrak{a}$ circles, also um $\mathfrak{a}(\mathfrak{a}-1)$, and consequently $\mathfrak{b}, \mathfrak{c}, \mathfrak{d}, \dots \mathfrak{y}, \mathfrak{z}$ concentric circles wll cut the plane into at most

$$(15) \qquad = \quad 2+2\mathfrak{V}+\mathfrak{a}(\mathfrak{a}-1)$$

regions, of which

$$(16) \qquad = \quad 1+2\mathfrak{V}+\mathfrak{a}(\mathfrak{a}-1)$$

regions are bounded.

8

From §3.5 we find that the plane will be cut into $1+U+A$ regions by $a, b, c, \dots y, z$ parallel lines. Were all of these lines intersected by $\mathfrak{a}$ concentric circles, the number of regions would increase by $2\mathfrak{a}(a+b+c+\cdots+z) = 2\mathfrak{a}U$, and by a second partitioning from $\mathfrak{b}$ concentric circles this number increases by $2\mathfrak{b}(\mathfrak{U}+\mathfrak{a})$, with a third partitioning from $\mathfrak{c}$ concentric circles by $2\mathfrak{c}(\mathfrak{U}+\mathfrak{a}+\mathfrak{I})$, and so on. In fact each circle increases the number of regions by the same number of points in which it intersects all the circles and lines that are present. It follows:

Rule 8.1: *That* $a, b, c, d, \dots y, z$ *parallel lines and* $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{d}, \dots \mathfrak{y}, \mathfrak{z}$ *concentric circles cut the plane into at most*

$$\begin{aligned}1+U+A &+ 2\mathfrak{a}U\\ &+ 2\mathfrak{a}(U+\mathfrak{a})\\ &+ 2\mathfrak{b}(U+\mathfrak{a}+\mathfrak{b})\\ &+ 2\mathfrak{c}(U+\mathfrak{a}+\mathfrak{b}+\mathfrak{c})\\ &+ 2\mathfrak{d}(U+\mathfrak{a}+\mathfrak{b}+\mathfrak{c})\\ &\vdots\\ &+ 2\mathfrak{z}(U+\mathfrak{a}+\mathfrak{b}+\mathfrak{c}+\cdots\mathfrak{y})\\ &= 1+U+A+2U\mathfrak{U}+2\mathfrak{A} \qquad (17)\end{aligned}$$

regions, of which (§3 and §6)

$$= 2U$$

are unbounded and therefore

$$= 1-U+A+2U\mathfrak{U}+2\mathfrak{A} \qquad (18)$$

are bounded, and whereby U and A are the sum and Amben of the numbers $a, b, c, \ldots z$ and $\mathfrak{U}$ and $\mathfrak{A}$ are the sum and Amben of the numbers $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \ldots \mathfrak{y}, \mathfrak{z}$.

9

In the above description connecting lines and circles (§8) both the a lines and the $\mathfrak{a}$ circles increase the number of regions by the number of intersection points of the lines with the circles, and therefore cut the plane into at most $\frac{a(a-1)}{1\cdot 2} + \mathfrak{a}(\mathfrak{a}-\mathfrak{a})$ regions, and therefore is follows:

"That $b, c, d, \ldots z$ parallel lines and a arbitrary lines, further $\mathfrak{b}, \mathfrak{c}, \mathfrak{b}, \ldots \mathfrak{y}, \mathfrak{z}$ concentric cicles and $\mathfrak{a}$ arbitrary circles together cut the plane into at most

$$= 1+U+A+2U\mathfrak{U}+2\mathfrak{A}+\frac{a(a-1)}{1\cdot 2}+\mathfrak{a}(\mathfrak{a}-1) \qquad (19)$$

regions, of which

$$= 2U \qquad (20)$$

are unbounded and

$$= 1-UA+2U\mathfrak{U}+2\mathfrak{A}+\frac{a(a-1)}{1\cdot 2}+\mathfrak{a}(\mathfrak{a}-1) \qquad (21)$$

bounded."

10

In the present connection between lines and circles, let $b = c = d = \cdots = z = 0$, and $\mathfrak{b} = \mathfrak{c} = \mathfrak{d} = \cdots .. = \mathfrak{z} = \mathfrak{o}$, and we find,

Rule 10.1: *with a arbitrary lines and $\mathfrak{a}$ arbitrary circles the plane can be cut into at most*

$$= 1+a+2a\mathfrak{a}+\frac{a(a-1)}{1\cdot 2}+\mathfrak{a}(\mathfrak{a}-1) \qquad (22)$$

regions of which

$$= 2a \qquad (23)$$

are unbounded, while

$$= 1-a+2a\mathfrak{a}+\frac{a(a-1)}{1\cdot 2}+\mathfrak{a}(\mathfrak{a}-1) \qquad (24)$$

are bounded.

It is easy to see that the equatiopns Eq. (11, 13, and 15) apply to the sphere just as the plane, in fact,

Rule 11.1: *Circles on the surface of a sphere can cut the surface into at most*

$$= 2 + \mathfrak{n}(\mathfrak{n} - \mathfrak{1}) \tag{25}$$

regions.

and furthermore:

Rule 11.2: *Using $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{d}, \cdots \mathfrak{z}$ concentric circles, the surface of a sphere can cut the surface into at most*

$$= 2 + 2\mathfrak{A} \tag{26}$$

regions, and with $\mathfrak{b}, \mathfrak{c}, \mathfrak{d}, \cdots, \mathfrak{z}$ concentric circles and $\mathfrak{a}$ arbitrary circle the surgface can be cut into at most

$$= 2 + 2\mathfrak{A} + \mathfrak{a}(\mathfrak{a} - \mathfrak{1}) \tag{27}$$

regions.

Rules on the partitioning of space using planes and spheres.

With a parallel planes, space is cut into $1 + a$ regions. With a second partitioning of b parallel planes, that cut the first a planes, the number of regions in space increases to $b(a + 1)$, with a third partioning from c parallel planes, also cutting the first sets of planes, the number of regions increases to $c(1 + a + b + ab)$, in fact with the
each of the c planes, the number of regions increases by just as much as the number of regions into which the existing planes are partitioned.

According to §3 and §5, each of the c planes will be cut into $1 + a + b + ab$ regions by the planes a and b, and consequently the number of regions will increase by as much. For the same reason the number of regions increases with the inclusion of d parallel planes intersecting the existing planes by $d(1 + a + b + c + ab + ac + bc)$, and so on. Therefore it follows that,

Rule 11.3: *The space containing $a, b, c, \ldots, y, z$ distinctly parallel planes will partion the space into at most*

$$\begin{aligned} 1+ \quad & + \quad a \\ & + \quad b(1+a) \\ & + \quad c(1+a+b+ab) \\ & + \quad d(1+a+b+c+ab+ac+hc) \\ & \qquad \vdots \\ & + \quad z(1+a+b+\cdots+y+ab+ac+\cdots ay+bc+\cdots+\cdots+xy) \\ & = \quad 1+U+A+T \end{aligned} \tag{28}$$

regions,

where U, A and T are the sum, amben, and ternen of the numbers $a, h, c, d, \ldots y, z$, i.e. the sum of the numbers, the sum of hte pairwise products of the numbers and sum of all distinct tripple products

In order to find out how many of these regions are unbounded and how many are bounded, consider the surface of a sphere for which att regions on the surface are bounded. This surface will be partioned into as many regions as there are unbounded regions of the planes cutting the surface. But since the planes divide the surface of the sphere into $2 + 2A$ regions (where A is the amben of the numbers $a, b, c, \cdots, z$), this is also the number of unbounded regions.

$$= 2 + 2A \tag{29}$$

and consequently the number of bounded regions, or the surface (by subtracting (29) from (28)):

$$= -1 + U - A + T \tag{30}$$

These two expressions can also be found by means similar to that used in deriving Eq. (28).

12

Assume that each of the partitions in §12 consist of only one plane, and the number of partions is n, that is to say assume $a = b = c \cdots = z = 1$ and likewise $a + b + c + \cdots z = n$, so obviously $U = n$, $A = \dfrac{n(n-1)}{1 \cdot 2}$ and $T = \dfrac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3}$
It therefore follows that,

Rule 12.1: *By Eq. (28) n arbitrary planes, space can be partitioned into at most*

$$= \quad 1 + n + \frac{n(n-1)}{1 \cdot 2} + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} \tag{31}$$

regions, and by Eq. (29)

$$= \quad 2 + n(n-1) \tag{32}$$

of these are unbounded and from Eq. (30)

$$\begin{aligned} &= \quad -1 + n - \frac{n(n-1)}{1 \cdot 2} + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} \\ &= \quad \frac{(n-1)(n-2)(n-3)}{1 \cdot 2 \cdot 3} \end{aligned} \tag{33}$$

are bounded.

As you can see from Eq. (33), no fewer than 4 planes can define a body (a finite region in space), and can define only one body. Furthermore the expression also shows that 5 planes can define 4, 6 planes can create 10, 7 planes 20, and for example 100 planes can define 156849 bounded regions, and this happens when no three planes are parallel with the same line (i.e. the planes are skew) and no four planes intersect at the same point.

13

Consider the partioning where only some sets consist of single planes, i.e. $q = r = s = \cdots = z = 1$ and $q + r + s + \cdots + z = m$ and denote the sum, Amben and Ternen including the remaining sets $a, b, c, d, \cdots p$, as U, A, und T, so that now $U = U_1 + m$, $A = A_1 + mU_1 + \dfrac{m(m-1)}{1 \cdot 2}$, and $T = T_1 + mA_1 + \dfrac{m(m-1)}{1 \cdot 2}U_1 + \dfrac{m(m-1)(m-2)}{1 \cdot 2 \cdot 3}$ from which it follows that

Rule 13.1: *With $a, b, c, \cdots$ sets of distinctly parallel planes, and through m arbitrary planes, a space, can be partitioned into at most*

$$= 1 + U_1 + A_1 + T_1 + \frac{m(m-1)}{1 \cdot 2}U_1 + mA_1 + m + \frac{m(m-1)}{1 \cdot 2} + \frac{m(m-1)(m-2)}{1 \cdot 2 \cdot 3} \tag{34}$$

regions, of which

$$= \quad 2 + 2A_1 + 2mU_1 + m(m-1) \tag{35}$$

are infinite, and

$$= \quad -1 + U_1 - A_1 + T_1 + \frac{m(m-3)}{1 \cdot 2}U_1 + m - \frac{m(m-1)}{1 \cdot 2} + \frac{m(m-1)(m-2)}{1 \cdot 2 \cdot 3} \tag{36}$$

are bounded.

14

Were the various cuts from $a, b, c, \cdots, z$ parallel planes (as in §12) to intersect $\mathfrak{a}$ concentric spherical surfaces, the number of regions increases to $\mathfrak{a}(2 + 2A)$. In fact the number increases by the same amount a the partitioning of each additional spherical surface cut by the planes, i.e. $2 + 2A$ (from §12). With a second partitioning from $\mathfrak{b}$ concetric spherical surfaces that intersect both the planes and the $\mathfrak{a}$ concentric surfaces, the number of regions increases to $\mathfrak{b}(2 + 2A + 2\mathfrak{a}U)$, because each of the $\mathfrak{b}$ surfaces partition the exisiting $\mathfrak{a}$ surfaces in accordance with Eq. (26), into $2 + 2A + 2\mathfrak{a}U$ regions, and therefore the number of regions increases accordingly. For the same reasons it follows that with a third partioning from $\mathfrak{c}$ concentric spherical surfaces intersecting the other surfaces, the number of regions increases by at most $\mathfrak{c}[2+2A+2(\mathfrak{a}+\mathfrak{b})U+\mathfrak{ab}]$. Likewise with a fourth partioning from $\mathfrak{d}$ concentric surfaces increases the nmber to $\mathfrak{d}[2+2A+2(\mathfrak{a}+\mathfrak{b}+\mathfrak{c})U+2\mathfrak{ab}+2\mathfrak{ac}+\mathfrak{2bc}]$, and so on. Therefore it follows that

Rule 14.1: *the space cut by $a, b, c, \cdots$ parallel planes, combinded with $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \cdots \mathfrak{z}$ concentric spherical surfaces is partioned into at most*

$$\begin{aligned}
1+U+A \quad &+ \quad T \ldots (\S.\ 12.) \\
&+ \quad \mathfrak{a}(2+2A) \\
&+ \quad \mathfrak{b}(2+2A+2\mathfrak{a}U) \\
&+ \quad \mathfrak{c}(2+2A+2(\mathfrak{a}+\mathfrak{b})U+2\mathfrak{ab}] \\
&+ \quad \mathfrak{d}[2+2A+2(\mathfrak{a}+\mathfrak{b}+\mathfrak{c})U+2\mathfrak{ab}+2\mathfrak{ac}+2\mathfrak{bc}] \\
&+ \quad \vdots \\
&+ \quad \mathfrak{z}[2+2A+2(\mathfrak{a}+\mathfrak{b}+\cdots+\mathfrak{z}))U+2\mathfrak{ab}+2\mathfrak{ac}+\cdots+2\mathfrak{yz}] \\
&= \quad 1+U+A+T+2\mathfrak{U}A+2\mathfrak{A}U+2\mathfrak{U}+2\mathfrak{T}
\end{aligned} \tag{37}$$

regions, from which (from §12),

$$= \quad 2+2A \tag{38}$$

are infinite and the remaining

$$= \quad 1+U-A+T+2\mathfrak{U}A+2\mathfrak{A}U+2\mathfrak{U}+2\mathfrak{T}t \tag{39}$$

are bounded.

where U, A, T stands for the sum, amben und ternen of the numbers $a, b, c, \cdots z$ and $\mathfrak{U}$, $\mathfrak{A}$, and $\mathfrak{T}$ represent the sum, amben and ternen of the numbers $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \cdots \mathfrak{z}$

15

From the general statement in §15, consider a specific partioning. Let $a=b=c=\cdots z=1$ and $a+b+c+\cdots+z=n$, likewise $\mathfrak{a}=\mathfrak{b}=\mathfrak{c}=\cdots=\mathfrak{z}=\mathfrak{1}$ and $\mathfrak{a}+\mathfrak{b}+\mathfrak{c}+\cdots+\mathfrak{z}=\mathfrak{n}$, then

$$U=n;\ A=\frac{n(n-1)}{1\cdot 2};\ T=\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}$$

and

$$\mathfrak{U}=\mathfrak{n};\ \mathfrak{A}=\frac{\mathfrak{n}(\mathfrak{n}-\mathfrak{1})}{\mathfrak{1}\cdot\mathfrak{2}};\ \mathfrak{T}=\frac{\mathfrak{n}(\mathfrak{n}-\mathfrak{1})(\mathfrak{n}-\mathfrak{2})}{\mathfrak{1}\cdot\mathfrak{2}\cdot\mathfrak{3}}.$$

It follows that

Rule 15.1: *Given n arbitrary planes, combined with n arbitrary spherical sufaces, the space can be divided into at most (see 37)*

$$\begin{aligned}
&= \quad 1+n+\frac{n(n-1)}{1\cdot 2}+\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}+\mathfrak{n}n(n-1)+n\mathfrak{n}(\mathfrak{n}-\mathfrak{1}) \\
&\quad\ 2\mathfrak{n}+2\frac{\mathfrak{n}(\mathfrak{n}-\mathfrak{1})\mathfrak{n}-\mathfrak{2})}{\mathfrak{1}\cdot\mathfrak{2}\cdot\mathfrak{3}}
\end{aligned} \tag{40}$$

regions, of which

$$= \quad 2+n(n-1) \tag{41}$$

are semi-infinite and

$$= \quad -1 + n - \frac{n(n-1)}{1 \cdot 2} + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} + \mathfrak{n}n(n-1)$$

(42) $$n\mathfrak{n}(\mathfrak{n}-1) + 2\mathfrak{n} + 2\frac{\mathfrak{n}(\mathfrak{n}-1)(\mathfrak{n}-2)}{1 \cdot 2 \cdot 3}$$

are bounded.

16

Reduce the number of planes to one is a portion of the sets, i.e. let $q = r = s \cdots = z = 1$ and $q + r + s + \cdots + z = m$, likewise $\mathfrak{q} = \mathfrak{r} = \mathfrak{s} = \cdots\ \mathfrak{z} = 1$ and $\mathfrak{q} + \mathfrak{r} + \mathfrak{s} + \cdots + \mathfrak{z} = \mathfrak{m}$, and again define the sum, Amben und ternen with the remaining numbers $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \cdots \mathfrak{p}$ by $\mathfrak{U}$, $\mathfrak{a}$, and $\mathfrak{T}$,

$$\begin{aligned} U \quad &= \quad U_1 + m;\ A = A_1 + mU_1 + \acute{a}\frac{m(m-1)}{1 \cdot 2};\ T = T_1 + mA_1 \\ &\quad + \frac{m(m-1)}{1 \cdot 2}U_1 + \frac{m(m-1)(m-2)}{1 \cdot 2 \cdot 3}, \\ \mathfrak{U} \quad &= \quad \mathfrak{U}_1 + \mathfrak{m};\ \mathfrak{A} = \mathfrak{A}_1 + \mathfrak{m}\mathfrak{U}_1 + \frac{\mathfrak{m}(\mathfrak{m}-1)}{1 \cdot 2};\ \mathfrak{T} = \mathfrak{T}_1 + \mathfrak{m}\mathfrak{A}_1 \\ &\quad \frac{m(m-1)}{1 \cdot 2}\mathfrak{U}_1 + \frac{m(m-1)(m-2)}{1 \cdot 2 \cdot 3}. \end{aligned}$$

Substitute these values into the folmula in §15, and it follows that

Rule 16.1: *The space cut by $a, b, c, \cdots p$ parallel planes and m arbitrary planes, combines with $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \cdots \mathfrak{p}$ concentric spherical surfaces and $\mathfrak{m}$ arbitrary spherical surfaces can be partioned into at most*

$$\begin{aligned} = \quad & 1 + U_1 + A_1 + T_1 + 2\mathfrak{U}_{1\,1} + 2\mathfrak{U}_1 U_1 + 2\mathfrak{U}_1 + 2\mathfrak{T}_1 \\ & + \frac{m(m-1)}{1 \cdot 2}\mathfrak{m}(\mathfrak{m}-1) + 2m\mathfrak{m})U_1 + (m + 2\mathfrak{m})A_1 + 2(m + \mathfrak{m})\mathfrak{U}_1 U_1 \end{aligned}$$

(43) $$+(m+\mathfrak{m})(m+\mathfrak{m}-1)\mathfrak{U}_1 + 2(m+\mathfrak{m})\mathfrak{A}_1 + m + \frac{m(m-1)}{1 \cdot 2} + \frac{m(m-1)(m-2)}{1 \cdot 2 \cdot}$$

(44) $$+m\mathfrak{m}(m+\mathfrak{m}-2) + 2\mathfrak{m} + 2\frac{\mathfrak{m}(\mathfrak{m}-1)(\mathfrak{m}-2)}{1 \cdot 2 \cdot 3}$$

regions of which

(45) $$= \quad 2 + 2A_1 + 2mU_1 + m(m-1)$$

are semi-infinite and

$$\begin{aligned} = \quad & -1 + U_1 = A_1 + T_1 + 2\mathfrak{U}_1 A_1 + 2\mathfrak{A}_1 U_1 + 2\mathfrak{U}_1 + 2\mathfrak{T}_1 \\ & + \left(\frac{m(m-1)}{1 \cdot 2} + \mathfrak{m}(\mathfrak{m}-1) + 2m\mathfrak{m}\right)U_1 + (m + 2\mathfrak{m})A_1 + 2(m+\mathfrak{m})\mathfrak{U}_1 U_1 \end{aligned}$$

(46) $$+(m+\mathfrak{m})(m+\mathfrak{m}-1)\mathfrak{U}_1 + 2(m+\mathfrak{m})\mathfrak{A}_1 + m - \frac{m(m-1)}{1 \cdot 2} + \frac{m(m-1)(m-2)}{1 \cdot 2 \cdot 3}$$

are bounded.

Or define the sum, amben und ternen of the numbers $a, b, c, \cdots p$ and the numbers m by U_2, A_2, T_2, likewise the numbers $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \cdots \mathfrak{p}$ and the number $\mathfrak{m}$ by $\mathfrak{U}_2, \mathfrak{A}_2, \mathfrak{T}_2$ then

$$\begin{aligned} U = U_2;\ A = A_2 + \frac{m(m-1)}{1 \cdot 2};\ T = T_2 + \frac{m(m-1)}{1 \cdot 2}U_2 - 2\frac{(m-1)m(m+1)}{1 \cdot 2 \cdot 3} \\ \mathfrak{U} = \mathfrak{U}_2;\ \mathfrak{A} = \mathfrak{A}_2 + \frac{\mathfrak{m}(\mathfrak{m}-1)}{1 \cdot 2};\ \mathfrak{T} = \mathfrak{T}_2 + \frac{\mathfrak{m}(\mathfrak{m}-1)}{1 \cdot 2}\mathfrak{U}_2 - 2\frac{(\mathfrak{m}-1)\mathfrak{m}(\mathfrak{m}+1)}{1 \cdot 2 \cdot 3} \end{aligned}$$

and Eq. (44, 45 and 46) become

$$
\begin{aligned}
&= 1 + U_2 + A_2 + T_2 + 2\mathfrak{U}_2 A_2 + 2\mathfrak{A}_2 U_2 + 2\mathfrak{U}_2 + 2\mathfrak{T}_2 \\
&\quad + \left(\frac{m(m-1)}{1\cdot 2} + \mathfrak{m}(\mathfrak{m}-1)\right) U_2 + (m(m-1) + \mathfrak{m}(\mathfrak{m}-1))\,\mathfrak{U}_2 \\
&\quad + \frac{m(m-1)}{1\cdot 2} - 2\frac{(m-1)m(m+1)}{1\cdot 2\cdot 3} - 4\frac{(\mathfrak{m}-1)\mathfrak{m}(\mathfrak{m}+1)}{1\cdot 2\cdot 3}. \qquad (47)
\end{aligned}
$$

$$= 2 + 2A_2 + m(m-1). \qquad (48)$$

$$
\begin{aligned}
&= -1 + U_2 - A_2 + T_2 + 2\mathfrak{U}_2 A_2 + 2\mathfrak{A}_2 U_2 + 2\mathfrak{U}_2 + 2\mathfrak{T}_2 \\
&\quad + \left(\frac{m(m-1)}{1\cdot 2} + \mathfrak{m}(\mathfrak{m}-1)\right) U_2 + (m(m-1) + \mathfrak{m}(\mathfrak{m}-1))\,\mathfrak{U}_2 \\
&\quad - \frac{m(m-1)}{1\cdot 2} - 2\frac{(m-1)m(m+1)}{1\cdot 2\cdot 3} - 4\frac{(\mathfrak{m}-1)\mathfrak{m}(\mathfrak{m}+1)}{1\cdot 2\cdot 3}. \qquad (49)
\end{aligned}
$$

17

Let $\mathfrak{a} = \mathfrak{b} = \mathfrak{c} = \cdots = \mathfrak{z} = 1$, and similarly $\mathfrak{a} + \mathfrak{b} + \mathfrak{c} + \cdots + \mathfrak{z} = \mathfrak{n}$, then $\mathfrak{U} = \mathfrak{n}$ and $\mathfrak{T} = \dfrac{\mathfrak{n}(\mathfrak{n}-1)\mathfrak{n}-2)}{1\cdot 2\cdot 3}$, and it follows from §18 that

Rule 17.1: *For* $\mathfrak{n}$ *arbitrary spherical surfaces, a space can be partitioned into at most*

$$= 2\mathfrak{n} + 2\frac{\mathfrak{n}(\mathfrak{n}-1)(\mathfrak{n}-2)}{1\cdot 2\cdot 3} \qquad (50)$$

regions, of which

$$= -1 + 2\mathfrak{n} + 2\frac{\mathfrak{n}(\mathfrak{n}-1)(\mathfrak{n}-2)}{1\cdot 2\cdot 3} \qquad (51)$$

are bounded.

18

Reduce the number of spherical surfaces in each set to one, so that $\mathfrak{q} = \mathfrak{r} = \mathfrak{s} = \cdots = \mathfrak{z} = 1$, and similarly $\mathfrak{q} + \mathfrak{r} + \mathfrak{s} + \cdots + \mathfrak{z} = \mathfrak{m}$, where as the sum, amben, und ternen of the numbers $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \cdots \mathfrak{p}$ given by $\mathfrak{U}_1$, $\mathfrak{A}_1$, und $\mathfrak{T}_1$,

$$\mathfrak{U} = \mathfrak{U}_1;\ \mathfrak{T} = \mathfrak{T}_1 + \mathfrak{m}\mathfrak{A}_1 + \frac{\mathfrak{m}(\mathfrak{m}-1)}{1\cdot 2}\mathfrak{U}_1 + \frac{\mathfrak{m}(\mathfrak{m}-1)(\mathfrak{m}-2)}{1\cdot 2\cdot 3},$$

and if follows from §18 that,

Rule 18.1: *The space cut by* $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \cdots \mathfrak{p}$ *concentric spherical surfaces , combined with* $\mathfrak{m}$ *arbitrary spherical surfaces can be partitioned into at most*

$$= 2\mathfrak{U}_1 + 2\mathfrak{T}_1 + \mathfrak{m}(\mathfrak{m}-\mathfrak{t})\mathfrak{U}_1 + 2\mathfrak{m}\mathfrak{A}_1 + 2\mathfrak{m} + 2\frac{\mathfrak{m}(\mathfrak{m}-1)(\mathfrak{m}-2)}{1\cdot 2\cdot 3} \qquad (52)$$

regions, of which

$$= -1 + 2\mathfrak{U}_1 + 2\mathfrak{T}_1 + \mathfrak{m}(\mathfrak{m}-\mathfrak{t})\mathfrak{U}_1 + 2\mathfrak{m}\mathfrak{A}_1 + 2\mathfrak{m} + 2\frac{\mathfrak{m}(\mathfrak{m}-1)(\mathfrak{m}-2)}{1\cdot 2\cdot 3} \qquad (53)$$

are bounded.

Or denote the sum and ternen of the numbers $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \cdots \mathfrak{p}$ and $\mathfrak{m}$ by $\mathfrak{U}_2$ and $\mathfrak{T}_2$ then, $\mathfrak{U} = \mathfrak{U}_2$ and $\mathfrak{T} = \mathfrak{T}_2 + \dfrac{\mathfrak{m}(\mathfrak{m}-1)}{1\cdot 2}\mathfrak{U}_2 - 2\dfrac{(\mathfrak{m}-1)\mathfrak{m}(\mathfrak{m}+1)}{1\cdot 2\cdot 3}$, which instead of Eq. (58, 59), yields the following:

$$= 2\mathfrak{U}_2 + 2\mathfrak{T}_2 + \mathfrak{m}(\mathfrak{m}-1)\mathfrak{U}_2 - 4\frac{(\mathfrak{m}-1)\mathfrak{m}(\mathfrak{m}+1)}{1\cdot 2\cdot 3}, \qquad (54)$$

$$= -1 + 2\mathfrak{U}_2 + 2\mathfrak{T}_2 + \mathfrak{m}(\mathfrak{m}-1)\mathfrak{U}_2 - 4\frac{(\mathfrak{m}-1)\mathfrak{m}(\mathfrak{m}+1)}{1\cdot 2\cdot 3}. \qquad (55)$$

19

Let $\mathfrak{a} = \mathfrak{b} = \mathfrak{c} = \cdots = \mathfrak{z} = 1$, and similarly $\mathfrak{a} + \mathfrak{b} + \mathfrak{c} + \cdots + \mathfrak{z} = \mathfrak{n}$, then $\mathfrak{U} = \mathfrak{n}$ and $\mathfrak{T} = \dfrac{\mathfrak{n}(\mathfrak{n}-1)\mathfrak{n}-2)}{1\cdot 2\cdot 3}$, and it follows from §18 that

Rule 19.1: *for $\mathfrak{n}$ arbitrary spherical surfaces, a space can be partitioned into at most*

$$= 2\mathfrak{n} + 2\frac{\mathfrak{n}(\mathfrak{n}-1)(\mathfrak{n}-2)}{1\cdot 2\cdot 3} \tag{56}$$

regions, of which

$$= -1 + 2\mathfrak{n} + 2\frac{\mathfrak{n}(\mathfrak{n}-1)(\mathfrak{n}-2)}{1\cdot 2\cdot 3} \tag{57}$$

are bounded.

20

Reduce the number of spherical surfaces in each set to one, so that $\mathfrak{q} = \mathfrak{r} = \mathfrak{s} = \cdots = \mathfrak{z} = 1$, and similarly $\mathfrak{q} + \mathfrak{r} + \mathfrak{s} + \cdots + \mathfrak{z} = \mathfrak{m}$, where as the sum, amben, und ternen of the numbers $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \cdots \mathfrak{p}$ given by $\mathfrak{U}_1$, $\mathfrak{A}_1$, und $\mathfrak{T}_1$,

$$\mathfrak{U} = \mathfrak{U}_1;\ \mathfrak{T} = \mathfrak{T}_1 + \mathfrak{m}\mathfrak{A}_1 + \frac{\mathfrak{m}(\mathfrak{m}-1)}{1\cdot 2}\mathfrak{U}_1 + \frac{\mathfrak{m}(\mathfrak{m}-1)(\mathfrak{m}-2)}{1\cdot 2\cdot 3},$$

and if follows from §18 that,

Rule 20.1: *The space cut by $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \cdots \mathfrak{p}$ concentric spherical surfaces , combined with $\mathfrak{m}$ arbitrary spherical surfaces can be partitioned into at most*

$$= \quad 2\mathfrak{U}_1 + 2\mathfrak{T}_1 + \mathfrak{m}(\mathfrak{m}-\mathfrak{t})\mathfrak{U}_1 + 2\mathfrak{m}\mathfrak{A}_1 + 2\mathfrak{m} + 2\frac{\mathfrak{m}(\mathfrak{m}-1)(\mathfrak{m}-2)}{1\cdot 2\cdot 3} \tag{58}$$

regions, of which

$$= \quad -1 + 2\mathfrak{U}_1 + 2\mathfrak{T}_1 + \mathfrak{m}(\mathfrak{m}-\mathfrak{t})\mathfrak{U}_1 + 2\mathfrak{m}\mathfrak{A}_1 + 2\mathfrak{m} + 2\frac{\mathfrak{m}(\mathfrak{m}-1)(\mathfrak{m}-2)}{1\cdot 2\cdot 3} \tag{59}$$

are bounded.

Or denote the sum and ternen of the numbers $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \cdots \mathfrak{p}$ and $\mathfrak{m}$ by $\mathfrak{U}_2$ and $\mathfrak{T}_2$ then, $\mathfrak{U} = \mathfrak{U}_2$ and $\mathfrak{T} = \mathfrak{T}_2 + \frac{\mathfrak{m}(\mathfrak{m}-1)}{1\cdot 2}\mathfrak{U}_2 - 2\frac{(\mathfrak{m}-1)\mathfrak{m}(\mathfrak{m}+1)}{1\cdot 2\cdot 3}$, which instead of Eq. (58, 59), yields the following:

$$= \quad 2\mathfrak{U}_2 + 2\mathfrak{T}_2 + \mathfrak{m}(\mathfrak{m}-1)\mathfrak{U}_2 - 4\frac{(\mathfrak{m}-1)\mathfrak{m}(\mathfrak{m}+1)}{1\cdot 2\cdot 3}, \tag{60}$$

$$= \quad -1 + 2\mathfrak{U}_2 + 2\mathfrak{T}_2 + \mathfrak{m}(\mathfrak{m}-1)\mathfrak{U}_2 - 4\frac{(\mathfrak{m}-1)\mathfrak{m}(\mathfrak{m}+1)}{1\cdot 2\cdot 3}. \tag{61}$$